$\mathcal{S}$-Normality and Property III in Linearly Ordered Extensions of Generalized Ordered Spaces

Masami HOSOBUCHI

Abstract

For a generalized ordered space X, we have two important linearly ordered extensions. One of them is X^* and the other one is $L(X)$. X^* contains X as a closed subspace, and was defined by D. J. Lutzer. $L(X)$ contains X as a dense subspace, and was defined by T. Miwa and N. Kemoto. $L(X)$ is called the minimal extension of X. Let X be an $\mathcal{S}$-normal GO-space or a GO-space with Property III. In this paper, we show that X^* and $L(X)$ have such properties under some countability conditions.

Key words : $\mathcal{S}$-normality, Property III, GO-space, linearly ordered extension, X^* , minimal extension.

1. Introduction

Let X be a linearly ordered set. Let us call X a linearly ordered topological space (LOTS) if X is given an order topology. In this case, the order topology is given by a subbase $\{]a, \rightarrow[,]\leftarrow, b[: a, b \in X\}$, where $]a, \rightarrow[= \{x \in X : a < x\}$, and similarly $]\leftarrow, b[= \{x \in X : x < b\}$. A LOTS is denoted by (X, λ) or $(X, \mathcal{T})$. A subspace of a LOTS is said to be a generalized ordered space (GO-space). A GO-space X inherits the order from a LOTS. We have another description: Let X be a linearly ordered set with a Hausdorff topology. Suppose that at each point, it is given a base of neighborhoods consisting of convex sets. The space with this topology becomes a GO-space. In the sequel, N denotes the set of natural numbers, and Z the set of integers. In Section 2, two linearly ordered extensions X^* and $L(X)$ are defined. It is well known that X is closed in X^* and that X is dense in $L(X)$. $\mathcal{S}$-normality and Property III are also defined in the section. Our theorems are stated in Section 3. The proofs are given in Section 4. Two remarks on the conditions in the theorems are given in the final section.

2. Definitions and theorems

Definition 2.1 Let (X, τ) be a GO-space, where τ denotes the topology on X. Two subsets R and L of X are defined as follows: $R=\{x\in X: [x, \rightarrow [\ \in \tau\setminus\lambda\}$ and $L=\{x\in X:]\leftarrow, x]\in\tau\setminus\lambda\}$, where λ denotes the order topology on X. A linearly ordered extension X^* that contains X as a closed subset of X^* is defined as a subset of $X\times Z$ as follows [5]:

$$X^* = X\times\{0\}\cup\{(x,-k): x\in R, k\in N\}\cup\{(x,k): x\in L, k\in N\}.$$

Let X^* have the lexicographic order. It is easily seen that $X\times\{0\}$ is a closed subspace of X^* and is identified with X. The minimal extention $L(X)$ of X is defined as a subset of a Cartesian product $X\times\{-1, 0, 1\}$ as follows [6]:

$$L(X)=X\times\{0\}\cup R\times\{-1\}\cup L\times\{1\}.$$

Let $L(X)$ have the lexicographic order topology. Then $X\times\{0\}$ is a subspace of $L(X)$ and is identified with X. It is easily seen that X is embedded densely in $L(X)$. Note that $L(X)$ is not a subspace of X^*.

Definition 2.2 A space X is called *$\mathcal{S}$-normal* if the following condition is satisfied: Let C be a closed subset of X. Then there exists a countable collection $\{U(n): n\in N\}$ of open subsets of X such that for $p\in C$ and $q\in X\setminus C$, there exists $n\in N$ such that $p\in U(n)$ and $q\notin U(n)$.

It is easy to see that a perfect space is $\mathcal{S}$-normal, where a space X is *perfect* if every closed subset of X is a G_δ-subset. Note that the Michael line is $\mathcal{S}$-normal and is not perfect. See [1] for studies of $\mathcal{S}$-normal GO-spaces. Among the results in [1] is "an $\mathcal{S}$-normal GO-space is hereditarily paracompact." Note that if X is an $\mathcal{S}$-normal GO-space, then it is first-countable.

Definition 2.3 A space X has *Property III* if and only if, for each $n\in N$, there are an open subset $U(n)$ of X and a relatively closed discrete subset $D(n)$ of $U(n)$ such that, for a point p and an open subset G of X that contains p, there exists an $n\in N$ such that $p\in U(n)$ and $G\cap D(n)\neq\emptyset$.

Property III was introduced by H. R. Bennett and D. J. Lutzer [2] to study GO-spaces with point-countable bases. They proved that a GO-space with Property III and a point-countable base is quasi-developable. It is shown in [2] that a GO-space having Property III is hereditarily paracompact. See [3], [4] for some results on Property III.

We state here our theorems.

Theorem 2.4 Let X be an *$\mathcal{S}$-normal GO-space. If the set $R\cup L$ defined in Definition 2.1 is countable, then X^* is $\mathcal{S}$-normal.*

THEOREM 2.5 *Let X be an $\mathcal{S}$-normal GO-space. Suppose that X satisfies the following conditions: For every $x \in R$, there exists an increasing sequence $\{x_n\}$ with $\sup\{x_n\}=x$ and, for $x \in L$, there exists a decreasing sequence $\{y_n\}$ with $\inf\{y_n\}=x$. If $R \cup L$ is countable, then $L(X)$ is $\mathcal{S}$-normal.*

THEOREM 2.6 *Let X be a GO-space having Property III. Then X^* has Property III.*

THEOREM 2.7 *Let X be a GO-space having Property III. If $R \cup L$ is countable, then $L(X)$ has Property III.*

3. Proofs of the theorems

To prove the theorems, we need technical preparation. For a convex open subset U of a GO-space X, we define a convex open subset $\tilde{U}$ of $E(X)$, where $E(X)$ denotes either X^* or $L(X)$. Then eight cases will occur. In the following, the intervals must be considered in $E(X)$.

(1) If a is the minimum point of U, then we define $\tilde{U}_1 = [(a, 0), \rightarrow [\subset E(X)$.
(2) Let $a = \inf U$, $a \notin U$. If $E(X)=L(X)$, then let $\tilde{U}_1 =](a, 1), \rightarrow [\subset L(X)$. Note that $(a, 1)$ may not exist in $L(X)$. If $E(X)=X^*$, then $U_1 =](a, +\infty), \rightarrow [\subset X^*$. The meaning of $(a, +\infty)$ should easily be understood.
(3) If there is a gap $u=(A, B)$ such that $U \cap B = U$, then we define $\tilde{U}_1 =](u, 0), \rightarrow [\subset E(X)$.
(4) If none of Cases 1－3 occur, then we define $\tilde{U}_1 = E(X)$.
(5) If b is the maximum point of U, then we define $\tilde{U}_2 =]\leftarrow, (b, 0)] \subset E(X)$.
(6) Let $b = \sup U$, $b \notin U$. If $E(X)=L(X)$, then let $\tilde{U}_2 =]\leftarrow, (b, -1)[\subset L(X)$. Note that $(b, -1)$ may not exist in $L(X)$. If $E(X)=X^*$, then let $\tilde{U}_2 =]\leftarrow, (b, -\infty)[\subset X^*$. The meaning of $(b, -\infty)$ also must be understood.
(7) If there is a gap $v=(A, B)$ such that $U \cap A = U$, then we define $\tilde{U}_2 =]\leftarrow, (v, 0)[\subset E(X)$.
(8) If none of Cases 5－7 occur, then we define $\tilde{U}_2 = E(X)$.

We set $\tilde{U} = \tilde{U}_1 \cap \tilde{U}_2$. $\tilde{U}$ is called the convex open set associated with U. Let U be an open set of a GO-space X. Then U is decomposed into a union of open convex subsets $\{U_\alpha : \alpha \in A\}$. In this case, we define $\tilde{U} = \cup \{\tilde{U}_\alpha : \alpha \in A\}$, where $\tilde{U}_\alpha$ is the convex open set associated with U_α. Then $\tilde{U}$ is an open subset of $E(X)$.

Proof of Theorem 2.4 Let $S = R \cup L = \{s_i : i \in N\}$ be an enumeration for $R \cup L$. Let C be a closed subset of X^*, $p \in C$ and $q \in X^* \setminus C$. We consider two cases.
Case 1. Assume that $C \cap (X \times \{0\}) = \emptyset$. For a point $(s_i, k) \in X^*$ with $s_i \in S$, let $U_+(s_i, k) =](s_i, k), \rightarrow [$ and $U_-(s_i, k) =] \leftarrow, (s_i, k)[$, where the intervals are considered in X^*. By the assumptions, we can write $p = (s_i, k)$ for some $s_i \in S$ and $k \in Z \setminus \{0\}$, where Z denotes the set of integers. If $q < p$, then it is clear that $p \in U_+(s_i, k-1)$ and $q \notin U_+(s_i, k-1)$. If $p < q$, then we have $p \in U_-(s_i, k+1)$ and $q \notin U_-(s_i, k+1)$. Hence $\{U_+(s_i, k), U_-(s_i, k) : s_i \in S, k \in Z, (s_i, k) \in X^*\}$ is a required collection of countable open subsets of X^*.

Case 2. Suppose that $C \cap X \neq \emptyset$, where X is identified with $X \times \{0\}$. Since $C \cap S$ is closed in X and since X is $\mathcal{S}$-normal, there exists a countable collection $\{U(n)\colon n \in N\}$ of open subsets of X such that, if $x \in C \cap X$ and $y \in X \setminus C$, then there exists $n \in N$ such that $x \in U(n)$ and $y \notin U(n)$. Then, a countable collection $\{\tilde{U}(n)\colon n \in N\} \cup \{U_+(s_i, k), U_-(s_i, k)\colon s_i \in S, k \in Z, (s_i, k) \in X^*\}$ of open subsets of X^* witnesses the $\mathcal{S}$-normality of X^*, where $\tilde{U}(n)$ denotes the open subset of X^* associated with $U(n)$ defined at the beginning of this section. To show this, we consider two cases. (1) Let $\pi(q) \in S$, where $\pi : X^* \to X$ denotes the projection to the first factor. Then $q = (s_i, k)$ for some $s_i \in S$ and $k \in Z$. If $q < p$, then we have $p \in U_+(s_i, k)$ and $q \notin U_+(s_i, k)$. If $p < q$, then we have $p \in U_-(s_i, k)$ and $q \notin U_-(s_i, k)$. (2) Suppose that $\pi(q) \notin S$. Let $E = X \setminus (R \cup L)$. Then it is clear that $q \in E \subset X \times \{0\}$. (i) Let $p = (s_i, k) \in X^*$ with $s_i \in S$ and $k \neq 0$. If $q < p$, then we have $p \in U_+(s_i, k-1)$ and $q \notin U_+(s_i, k-1)$. If $p < q$, then we have $p \in U_-(s_i, k+1)$ and $q \notin U_-(s_i, k+1)$. (ii) Let $p \in E \subset X \times \{0\}$ or $p = (s_i, 0)$ with $s_i \in S$. Then $p \in C \cap X$ and $q \in E \setminus C$. Hence there exists $n \in N$ such that $p \in U(n)$ and $q \notin U(n)$. Thus $p \in \tilde{U}(n)$ and $q \notin \tilde{U}(n)$. This completes the proof.

Proof of Theorem 2.5 We use the notation in the proof of Theorem 2.4. Let C be a closed subset of $L(X)$, $p \in C$ and $q \in L(X) \setminus$ C. We shall consider two cases.
Case 1. Let $C \cap X = \emptyset$. Let $z = (s_i, \varepsilon)$ be a point of $L(X) \cap (X \times \{-1, 1\})$. By the assumption of the theorem, there exist two monotone sequences $\{x_k(s_i)\}$ and $\{y_k(s_i)\}$ such that $s_i = \sup\{x_k(s_i)\} = \inf\{y_k(s_i)\}$. If $\varepsilon = -1$, we set $V_-(s_i, k) =]\, x_k(s_i), z]$, an open interval in $L(X)$. For $\varepsilon = 1$, then we set $V_+(s_i, k) = [z, x_k(s_i)[$. Now let $p \in C$ and $q \in L(X) \setminus C$. If $p = (s_i, -1)$ and $p < q$, then we have $p \in V_-(s_i, 1)$ and $q \notin V_-(s_i, 1)$. Let $p = (s_i, -1)$ and $q < p$. Since $\pi(q) < s_i$, there exists $x_k(s_i) \in X$ such that $\pi(q) < x_k(s_i) < s_i$, where $\pi : L(X) \to X$ is the projection to the first factor. Hence $p \in V_-(s_i, k)$ and $q \notin V_-(s_i, k)$. For the case of $p = (s_i, 1)$, the proof is done analogously. Hence $\{V_+(s_i, k), V_-(s_i, k)\colon s_i \in S, k \in N\}$ assures the $\mathcal{S}$-normality of $L(X)$.
Case 2. Suppose that $C \cap X \neq \emptyset$. Since $C \cap X$ is closed in X, there exists a collection of countable open subsets of X $\{U(n)\colon n \in N\}$ such that, if $x \in C \cap X$ and $y \in X \setminus C$, then there exists $n \in N$ such that $x \in U(n)$ and $y \notin U(n)$. Then, a countable collection $\{\tilde{U}(n)\colon n \in N\} \cup \{V_+(s_i, k), V_-(s_i, k)\colon s_i \in S, k \in N\} \cup \{U_+(s_i, \varepsilon), U_-(s_i, \varepsilon)\colon s_i \in S, \varepsilon = \pm 1, (s_i, \varepsilon) \in L(X)\}$ of open subsets of $L(X)$ guarantees the $\mathcal{S}$-normality of $L(X)$. To prove this, we consider three cases. (1) Let $\pi(q) \in S$. Then $q = (s_i, \varepsilon)$ for some $s_i \in S$ and $\varepsilon \in \{-1, 0, 1\}$. If $q < p$, then $p \in U_+(s_i, \varepsilon)$ and $q \notin U_+(s_i, \varepsilon)$. If $p < q$, then $p \in U_-(s_i, \varepsilon)$ and $q \notin U_-(s_i, \varepsilon)$. (2) Let $\pi(q) \notin S$. Then $q \in E \subset X \times \{0\}$ as shown in the proof of Theorem 2.4. If $p = (s_i, \varepsilon)$, where $\varepsilon \in \{-1, 1\}$, then the proof is similar to Case 1. (3) Let $p \in E \subset X \times \{0\}$ or $p = (s_i, 0)$ with $s_i \in S$. Then we have $p \in C \cap E$ and $q \in E \setminus$ C. There exists $n \in N$ such that $p \in U(n)$ and $q \notin U(n)$. Hence $p \in \tilde{U}(n)$ and $q \notin \tilde{U}(n)$. This completes the proof.

Proof of Theorem 2.6 Let X be a GO-space having Property III. Let $\{U(n), D(n)\colon n \in N\}$ be the countable collection given in Definition 2.3 that guarantees Property III of X. Let $\tilde{U}(n)$ be the open subset of X^* associated with $U(n)$. Then it is easy to see that $D(n)$ is relatively closed discrete in $\tilde{U}(n)$. Set $\tilde{U}(0) = \{x \in X^*\colon \{x\}$ is open in $X^*\}$ and $D(0) = \tilde{U}(0)$. Then the countable collection $\{\tilde{U}(n), D(n)\colon n \geq 0\}$ will guarantee Property III of X^*. To prove this, let $p \in G$, where G is open in X^*. If p

is isolated, we take $\tilde{U}(0)$ for which we have $p \in U(n)$ and $G \cap D(0) \neq \emptyset$. Let p be non-isolated. Then $p \in X \times \{0\}$. Since $G \cap X$ is an open set of X that contains p, there exists $n \in N$ such that $p \in U(n)$ and $D(n) \cap G \cap X \neq \emptyset$. It is clear that $p \in \tilde{U}(n)$ and $G \cap D(n) \neq \emptyset$ by the construction of $\tilde{U}(n)$.

Proof of Theorem 2.7 Let X be a GO-space having Property III. Therefore, there exists a family $\{U(n), D(n)\}$ that guarantees the property. Let $S = R \cup L = \{s_i : i \in N\}$ be the enumeration for $R \cup L$. We define a countable collection $\{L(s_i, \varepsilon), P(s_i, \varepsilon) : s_i \in S, (s_i, \varepsilon) \in L(X)\} \cup \{U(n, k), D(n, k) : n \in N, k \in N\}$ that will guarantee Property III of $L(X)$. For $(s_i, \varepsilon) \in L(X)$ with $s_i \in S$, let $L(s_i, \varepsilon) = L(X)$ and $P(s_i, \varepsilon) = \{(s_i, \varepsilon)\}$. Take $U(n) \subset X$ stated above and consider a convex subset W of $U(n) \setminus S$ that is open and maximal in X, where the convexity is considered in X, and the maximality is in the sense that if W' contains W as a proper subset, then $W' \cap S = \emptyset$. Such W's are countably many and we enumerate such subsets as $\{U(n, k) : k \in N\}$. Let $E = X \setminus (R \cup L)$. It is clear that $U(n, k)$ is open in $L(X)$, because $U(n, k) \subset E$. We define $D(n, k) = D(n) \cap U(n, k)$. It is obvious that $D(n, k)$ is closed discrete in $U(n, k)$. Then we can show that the countable collection

$$\{L(s_i, \varepsilon), P(s_i, \varepsilon) : s_i \in S, (s_i, \varepsilon) \in L(X)\} \cup \{U(n, k), D(n, k) : n \in N, k \in N\}$$

guarantees Property III of $L(X)$. To prove this, let $p \in G$, where G is a convex open subset of $L(X)$. If $p = (s_i, \varepsilon)$ for some $s_i \in S$, then it is clear that $p \in L(s_i, \varepsilon)$ and $G \cap P(s_i, \varepsilon) \neq \emptyset$. Let $p \in X \setminus S$. Then $p \in E \subset X$. If $G \cap X$ contains a point of S, say s_i, it is obvious that $p \in L(s_i, 0)$ and $G \cap P(s_i, 0) \neq \emptyset$. Hence we may assume that $G \cap X \cap S = \emptyset$. Therefore, $p \in G \cap X \subset E$. Since $G \cap X$ is an open set of X that contains p, there exists $n \in N$ such that $p \in U(n)$ and $D(n) \cap G \cap X \neq \emptyset$. Since $p \in U(n) \cap G \cap X \subset E$ and $G \cap X$ is convex in X, there exists $k \in N$ such that $p \in U(n, k)$ and $D(n, k) \cap G \cap X \neq \emptyset$. Therefore, it follows that $p \in U(n, k)$ and $D(n, k) \cap G \neq \emptyset$. This completes the proof.

4. Remarks

Remark 4.1 Theorems 2.4, 2.5 and 2.7 do not necessarily hold if we drop the assumption of the countability of $S = R \cup L$ as the following examples show: (i) Let ω_1 be the set of countable ordinals with the usual order. This is a LOTS, but it is not paracompact. Consider $X = (\omega_1, \text{discrete})$, a GO-space on ω_1 with the discrete topology. Since X is metrizable, X is $\mathcal{S}$-normal and has Property III. In this case, R is the set of limit ordinals of ω_1. Therefore, $|R| > \omega$. Since $L(X)$ is homeomorphic to ω_1, $L(X)$ is neither $\mathcal{S}$-normal nor does it have Property III by the comments after Definitions 2.2 and 2.3. (ii) Let $\mathbb{S} = (\mathbb{R}, \mathcal{S})$ be the Sorgenfrey line. Since $\mathbb{S}$ is perfect, it is $\mathcal{S}$-normal. Since $R = \mathbb{R}$, it is not a countable set. Then X^* is not $\mathcal{S}$-normal. To show this, let $C = \{(x, 0) : x \in \mathbb{R}\}$. Then C is a closed subset of X^*. However, there does not exist a countable collection of open subsets of X^* that separates points $(p, 0)$ of C and points $(p, -1) \in X^* \setminus C$.

Remark 4.2 There are no relations between $\mathcal{S}$-normality and Property III as the following examples show. (i) Let X be the Souslin line with a point-countable base. It is consistent that such space exists. It is known that it is perfect, hence $\mathcal{S}$-normal. However, it does not have Property III, because of Theorem 1.5 in [2]. (ii) Let X be the unit square I^2 with the lexicographic order-topology. It is shown that I^2 has Property III in [2]. However, this is not $\mathcal{S}$-normal. To show this, let $C = I \times \{0, 1\}$. It is obvious that C is a closed subset of I^2. Then there is no countable collection of open subsets of I^2 that separates points of $I \times \{1\}$ and points of $I \times \{1/2\}$.

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Tokyo Kasei Gakuin University
Department of Housing and Planning
2600 Aihara, Machida, Tokyo 194-0292
JAPAN
E-mail: mhsbc@kasei-gakuin.ac.jp