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Relationships between four properties of topological spaces and perfectness of generalized ordered spaces

Masami HOSOBUCHI

Abstract

In this paper, we investigate relationships between four properties of topological spaces that were defined by H. R. Bennett, D. J. Lutzer and S. D. Purisch in [BLP]. Among their results is a theorem: A GO-space that has a σ-closed discrete dense subset has a dense metrizable subspace. We can generalize this as follows: A first-countable, collectionwise Hausdorff, regular space has a dense metrizable subspace. Furthermore, we state a theorem concerning perfectness of generalized ordered spaces.

Key words : Properties I, II, III and IV, relatively discrete, collectionwise Hausdorff, GO-space, F_σ-subset, perfect.

1. Four properties

The following definitions were given in [BLP].

Definition 1. A topological space X is said to have Property I if and only if there exists a σ-closed discrete dense subset D of X, that is, $D = \cup \{D(n) : n \in N\}$ is a dense subset of X such that $D(n)$ is a closed discrete subset of X. N denotes the set of natural numbers.

Definition 2. A topological space X is said to have Property II if and only if there is a dense metrizable subspace of X.

Definition 3. A topological space X is said to have Property III if and only if, for each $n \in N$, there are an open subset $U(n)$ of X and a relatively closed discrete subset $D(n)$ of $U(n)$ such that, for a point p and an open subset G of X that contains p, there exists an $n \in N$ such that $p \in U(n)$ and $G \cap D(n) \neq \phi$ (see also [BL2], [H]).

Definition 4. A topological space X is said to have Property IV if and only if there exists a σ-relatively discrete dense subset D of X, that is, $D = \cup \{D(n) : n \in N\}$ is a dense subset of X such that $D(n)$ is a relatively discrete subspace. "Relatively discrete" means "discrete as a subspace".

We begin this section by showing the following proposition (see also [BLP]).

Proposition 1. *For any topological space, Property I implies Property III. Property II implies Property IV. Property III implies Property IV.*

Proof. We show that Property I implies Property III. Let X be a topological space having Property I. Let $D = \cup D(n)$ be a σ-closed discrete dense subset D of X. Let $U(n) = X$ for every $n \in N$. Then $\{U(n), D(n) : n \in N\}$ witnesses Property III. To see this, let G be an open subset of X containing p. Since D is a dense subset of X, $G \cap D \neq \phi$. Hence $G \cap D(n) \neq \phi$ for some $n \in N$. Since $U(n) = X$, $p \in U(n)$. We show that Property II implies Property IV. Let D be a dense metrizable subspace of X. Since every metrizable space has a σ-discrete base [E, p.281], D has a σ-discrete base $\boldsymbol{B} = \cup \{\boldsymbol{B}(n) : n \in N\}$ such that $\boldsymbol{B}(n)$ is a discrete family in D for each $n \in N$. Fix $n \in N$. For every element B of $\boldsymbol{B}(n)$, take a point $d(B) \in B$. Then it is clear that $D(n) = \{d(B) : B \in \boldsymbol{B}(n)\}$ is a relatively discrete subset of D. We show that $D' = \cup \{D(n) : n \in N\}$ is a dense subset of D. To see this, let G be an open neighborhood of d in D. Since $\boldsymbol{B} = \cup \{\boldsymbol{B}(n) : n \in N\}$ is a base for D, there exist an $n \in N$ and an element $B \in \boldsymbol{B}(n)$ such that $d \in B \subset G$. Since $d(B) \in B$, $d(B) \in D(n) \cap G$ by the definition of $D(n)$. Hence $D' \cap G \neq \phi$. Since D is a dense subset of X, so is D'. Therefore, $D' = \cup \{D(n) : n \in N\}$ is a σ-relatively discrete dense subset of X. We show that Property III implies Property IV. Let $\{U(n), D(n) : n \in N\}$ be a family of open subsets and relatively closed subsets which is guaranteed in the definition of Property III. Then it is easy to see that $D = \cup \{D(n) : n \in N\}$ is a σ-relatively discrete dense subset of X. This completes the proof.

2. A sufficient condition for which Property I implies Property II

In [BLP], H. R. Bennett, D. J. Lutzer and S. D. Purisch proved that a GO-space having Property I has Property II. We can generalize it as Theorems 1 and 2 below. The topological space that appears in Theorem 1 is not assumed to be a GO-space, but we need to assume that the space is regular, first-countable and collectionwise Hausdorff.

Definition 5. A linearly ordered topological space $(X, \lambda, <)$ is a linearly ordered set $(X, <)$ with an interval topology λ. A basic open set in the interval topology is an open interval $]a, b[$, where a, b are points of X and $]a, b[= \{x \in X : a < x < b\}$. A GO-space or a generalized ordered space is a subspace of a linearly ordered topological space.

Definition 6. A topological space X is called collecionwise Hausdorff if and only if, for any closed discrete subset E of X, there exists a family $W = \{W(e) : e \in E\}$ of open subsets of X such that $e \in W(e)$ and W is discrete in X, where W is discrete in X if and only if for any $x \in X$, there exists a neighborhood $V(x)$ of x such that $\# \{W(e) : V(x) \cap W(e) \neq \phi\} \leqq 1$.

Theorem 1. *Let X be a regular space having Property I. If X is first-countable and collectionwise Hausdorff, then X has Property II.*

Proof. Let $D = \cup \{D(n) : n \in N\}$ be a σ-closed discrete dense subset of X, where $D(n)$ is a closed discrete subset of X for each $n \in N$. Since X is collectionwise Hausdorff, there exists a discrete family $W(n) = \{W(d\,;\,n) : d \in D(n)\}$ of open subsets of X for each $n \in N$ such that $d \in W(d\,;\,n)$. Let $d \in D(n)$. Since X is first-countable, there exists a countable neighborhood base $\{U(d\,;\,n, m) : m \in N\}$ at d. Let $V(d\,;\,n, m) = U(d\,;\,n, m) \cap W(d\,;\,n)$. Then $V(d\,;\,n, m)$ is an open neighborhood of d and is contained in $W(d\,;\,n)$. Let $\boldsymbol{B}(n, m) = \{V(d\,;\,n, m) \cap D : d \in D(n)\}$. Then $B(n, m)$ is a discrete family in D. To see this, it is enough to note that $\{W(d\,;\,n) : d \in D(n)\}$ is a discrete family in X and $V(d\,;\,n, m) \cap D \subset W(d\,;\,n)$. We show that $B = \cup \cup \{B(n, m) : n \in N, m \in N\}$ is a base for D. Let $d \in D$ and G be an open subset of D that contains d. Then $d \in D(n)$ for some $n \in N$ and there is an open subset G' of X such that $G' \cap D = G$. Since $\{U(d\,;\,n, m) : m \in N\}$ is a countable neighborhood base at d, there exists an $m \in N$ such that $U(d\,;\,n, m) \subset G'$. Hence $V(d\,;\,n, m) \cap D = U(d\,;\,n, m) \cap W(d\,;\,n) \cap D \subset U(d\,;\,n, m) \cap D \subset G' \cap D = G$. Therefore, D has a σ- discrete base. By Bing Metrization Theorem [E, p.282], D is metrizable. Hence X has Property II. This completes the proof.

In [BLP], they used a concept “semistratifiable” to show that a GO-space having Property I has Property II. We see that the result (Theorem 2, below) can be shown directly and is obtained as a corollary to Theorem 1. Hence Theorem 1 is a generalization of the following theorem proved in [BLP].

Theorem 2 [BLP]. *A GO-space X with Property I has Property II.*

Proof. We prove this theorem by making use of Theorem 1. It is sufficient to show that X is regular, first-countable and collectionwise Hausdorff by virtue of Theorem 1. Since a GO-space is collectionwise normal, it is clear that X is regular and collectionwise Hausdorff. Hence it is enough to prove that X is first-countable. Let $p \in X$. Suppose that X is not first-countable at p. First we assume that $[p, \rightarrow[$ is not an open set of X. Let κ be the cofinality of the set $]\leftarrow, p[$. Then we can take a strictly increasing net $\{x(\alpha) : \alpha < \kappa\} \subset]\leftarrow, p[$ that converges to p. If κ is countable, then it is clear that there exists a strictly increasing sequence $\{x(n) : n \in N\}$ in $]\leftarrow, p[$ that converges to p. We assume that κ is uncountable. Since X has Property I, there exists a σ-closed discrete dense subset $D = \cup \{D(n) : n \in N\}$ of X. For each $\alpha < \kappa$, there is a point $d \in]\leftarrow, p[\cap D$ such that $x(\alpha) < d$ since $]x(\alpha), p[$ is a non-empty open set of X and since D is a dense subset of X. Since d belongs to $D(n)$ for some $n \in N$, we write $d = d(n(\alpha), \alpha)$. Since κ is uncountable, there exists an uncountable cofinal subset $A \subset \kappa$ and an $m \in N$ such that $n(\alpha) = m$ for every $\alpha \in A$. Then $\{d(m, \alpha) : \alpha \in A\}$ is a subset of $D(m)$ that converges to p. This is a contradiction because $D(m)$ is a closed discrete subset of X. Hence there exists a strictly increasing sequence $\{x(n) : n \in N\}$ that converges to p. If $]\leftarrow, p]$ is not open in X, then we can take a strictly decreasing sequence $\{y(n) : n \in N\}$ in $]p, \rightarrow[$ which converges to p by a parallel discussion to the above arguments. There are four cases to consider.

Case 1. Neither $[p, \rightarrow[$ nor $]\leftarrow, p]$ are open in X. Then a collection $\{]x(n), y(n)[: n \in N\}$ is a countable neighborhood base at p.

Case 2. $[p, \rightarrow[$ is not open in X, but $]\leftarrow, p]$ is open in X. Then a family $\{]x(n), p] : n \in N\}$ is a countable neighborhood base at p.

Case 3. $[p, \rightarrow[$ is open in X, but $]\leftarrow, p]$ is not open in X. Then a collection $\{[p, y(n)[: n \in N\}$ is a countable neighborhood base at p.

Case 4. Both $[p, \rightarrow[$ and $]\leftarrow, p]$ are open in X. Then $\{p\}$ is open in X.

Hence X is first-countable. This completes the proof.

3. Converse questions

In this section, we consider topological spaces that have Property IV, that is, spaces that have σ - relatively discrete dense subsets. When do such spaces have other three properties defined in the first section?

Definition 7. A topological space X is perfect if every open subset of X is an F_σ-subset of X. An F_σ-subset of X is a union of countably many closed subsets of X.

In [BLP], it was proved that all four properties defined in Section 1 are equivalent for perfect GO-spaces. The following is pointed out in [BLP] without proof.

Proposition 2. *Let X be a topological space having Property IV. If X is perfect, then X has Property I.*

Proof. Let X be a perfect topological space having Property IV. Let $D = \cup \{D(n) : n \in N\}$ be a σ - relatively discrete dense subset of X, where D is a dense subset and $D(n)$ is relatively discrete for every $n \in N$. Fix $n \in N$. For each $d \in D(n)$, we choose a neighborhood $W(d, n)$ of d such that $W(d, n) \cap D(n) = \{d\}$. Let $W(n) = \cup \{W(d, n) : d \in D(n)\}$. Since X is perfect and $W(n)$ is an open subset of X, $W(n)$ is expressed as $W(n) = \cup \{F(n, m) : m \in N\}$, where $F(n, m)$ is a closed subset of X for every $m \in N$. Let $D(n, m) = F(n, m) \cap D(n)$. Then $D(n, m)$ is a closed discrete subset of X. It is clear that $D(n, m)$ is relatively discrete since $D(n, m) \subset D(n)$. To see that $D(n, m)$ is closed in X, let $p \in X - D(n, m)$. If p does not belong to $F(n, m)$, then there exists a neighborhood $V(p)$ of p such that $V(p) \cap F(n, m) = \phi$ since $F(n, m)$ is closed in X. It is clear that $V(n) \cap D(n, m) = \phi$ since $D(n, m) \subset F(n, m)$. Suppose that $p \in F(n, m) - D(n, m)$. Since $p \in F(n, m) \subset W(n) =$

$\cup$ $\{W(d, n) : d \in D(n)\}$, there exists a point $d \in D(n)$ such that $p \in W(d, n)$. Then $p \neq d$. To see this, suppose that $p = d$. Since $p \in F(n, m)$ and $d \in D(n)$, $p \in D(n, m)$. But, this is a contrdiction because p does not belong to $D(n, m)$. Hence $W(d, n) - \{d\}$ is a neighborhood of p. It is obvious that $(W(d, n) - \{d\}) \cap D(n, m) = \phi$ since $(W(d, n) - \{d\}) \cap D(n) = \phi$. Therefore, $D(n, m)$ is a closed subset of X. It is easy to see that $D = \cup\cup \{D(n, m) : n \in N, m \in N\}$. Hence D is a σ- closed discrete dense subset of X and hence X has Property I.

Corollary 1. *Let X be a perfect, regular space having Property IV. If X is first-countable and collectionwise Hausdorff, then X has a dense metrizable subspace.*

Proof. By Proposition 2, X has Property I. Hence X has Property II by virtue of Theorem 1. Property II means the existence of a dense metrizable subspace of X.

4. Perfectness of GO-spaces

In this section, we state a theorem concerning perfectness of a GO-space and its dense subspace which was obtained in [BHL]. As its proof invoked a lemma : A GO-space X is perfect if and only if every relatively discrete subset of X is an F_σ-subset of X, we give a direct proof of the theorem in this paper. At the same time, we state results in its generalized forms. This clarifies the differences between two proofs. We begin by showing the following lemma.

Lemma 1. *Let X be an arbitrary topological space. Let* $\{U(\alpha) : \alpha < \kappa\}$ *be a collection of open subsets of X and* $Z = \{z(\alpha) : \alpha < \kappa\}$ *a subset of X such that* $z(\alpha) \in U(\alpha)$. *If* $U(\alpha) \cap Z = \{z(\alpha)\}$ *and* $U = \cup \{U(\alpha) : \alpha < \kappa\}$ *is an* F_σ*-subset of X, then* $Z = \{z(\alpha) : \alpha < \kappa\}$ *is an* F_σ*- subset of X.*

Proof. Since U is an F_σ-subset of X, $U = \cup \{F(n) : n \in N\}$, where $F(n)$ is closed in X. Set $Z(n) = F(n) \cap Z$. Then $Z(n) \subset F(n) \subset U$. To show that $Z(n)$ is closed in X, let x be a point of X that does not belong to $Z(n)$. Suppose that x does not belong to $F(n)$. Then there exists a neighborhood $V(x)$ of x such that $V(x) \cap F(n) = \phi$ since $F(n)$ is closed in X. Hence $V(x) \cap Z(n) = \phi$ since $Z(n) \subset F(n)$. If $x \in F(n) - Z(n)$, then there exists $\alpha < \kappa$ such that $x \in U(\alpha)$ since $U = \cup \{F(n) : n \in N\} = \cup \{U(\alpha) : \alpha < \kappa\}$. Since, as is easily seen, $x \neq z(\alpha)$ and $U(\alpha) \cap Z = \{z(\alpha)\}$, it follows that $U(\alpha) - \{z(\alpha)\}$ is a neighborhood of x and $(U(\alpha) - \{z(\alpha)\}) \cap Z(n) = \phi$. It is obvious that $Z = \cup \{Z(n) : n \in N\}$. Hence Z is an F_σ-subset of X.

Corollary 2. *Suppose that X is a perfect topological space. Every relatively discrete subspace of X is an* F_σ*-subset of X.*

Proof. Let $Z = \{z(\alpha) : \alpha < \kappa\}$ be a relatively discrete subspace of X. Then, for each $\alpha < \kappa$, we can take a neighborhood $U(z(\alpha))$ of $z(\alpha)$ such that $U(z(\alpha)) \cap Z = \{z(\alpha)\}$. Since X is perfect, $U =$

$\cup \{U(z(\alpha)) : \alpha < \kappa\}$ is an F_σ-subset of X. It follows from Lemma 1 that Z is an F_σ-subset of X.

We need the following two lemmas to show Theorem 3.

Lemma 2. *Let Y be a GO-space. If there exists a dense subspace X of Y such that the following condition is satisfied, then Y is first-countable : Every relatively discrete subspace of X is an F_σ-subset of Y.*

Proof. Let y be an arbitrary point of Y. First we consider a half line $H =]\leftarrow, y]$ in Y and show that the first-countability at y in H. If y is not a limit point, that is, if $[y, \rightarrow[$ is not open in Y, then there is nothing to prove. Let y be a limit point of the set $]\leftarrow, y[$ and let κ be the cofinality of $]\leftarrow, y[$. If κ is countable, then it is clear that there exists a strictly increasing sequence $\{x(n)\}$ in $]\leftarrow, y]$ which converges to y. Suppose that κ is uncountable. Since X is dense in Y, there exists a strictly increasing net $\{x(\alpha) : \alpha < \kappa\}$ in $X \cap]\leftarrow, y[$ such that $y = \lim \{x(\alpha) : \alpha < \kappa\}$. Let $Z = \{x(\alpha) : \alpha$ is isolated in $\kappa\}$, then Z is clearly a relatively discrete subset of X. By the assumption, we can write $Z = \cup \{Z(n) : n \in N\}$, where $Z(n)$ is a closed subset of Y. Note that "$\sup Z(n) = y$ " does not occur for every $n \in N$. To see this, suppose that $\sup Z(n_0) = y$ for some $n_0 \in N$. Because y does not belong to $Z(n_0)$, then $Z(n_0)$ is not closed. This is a contradiction. Let $s(n) = \sup Z(n)$, then $s(n) < y$. We may assume that $s(n) < s(n+1)$ for each $n \in N$ since $Z = \cup \{Z(n) : n \in N\}$ and since $\sup Z = y$. Then y is a limit point of a strictly increasing sequence $\{s(n) : n \in N\}$ in X. If $]\leftarrow, y]$ is open in Y, a collection $\{]s(n), y] : n \in N\}$ is a countable neighborhood base at y. If $]\leftarrow, y]$ is not open in Y, then consider the half line $[y, \rightarrow[$. By the parallel observation, there exists a strictly decreasing sequence $\{t(n) : n \in N\}$ such that y is its limit point. Then a family $\{]s(n), t(n)[: n \in N\}$ is a countable neighborhood base at y. If $[y, \rightarrow[$ is open in Y and $]\leftarrow, y]$ is not open in Y, then a collection $\{[y, t(n)[: n \in N\}$ is a countable neighborhood base at y. Hence Y is first-countable.

Lemma 3. *Let Y be a GO-space. Assume that there exists a dense subspace X of Y that satisfies the following condition : Every relatively discrete subspace of X is an F_σ-subset of Y. If J is a convex subset of Y, then J is written as $J = \cup \{F(n) : n \in N\}$, where*
(1) $F(n)$ is closed, convex in Y for each $n \in N$,
(2) $F(1) \subset F(2) \subset \ldots \subset F(n) \subset \ldots$.

Proof. We may assume that $J =]p, q[$, where p and q are limit points of J, because if $J = [p, q[$ or $]p, q]$ (or even the case p (or q) is a gap), then the proof is easier than the case $J =]p, q[$. If $J = [p, q]$, then there is nothing to prove. By Lemma 2, there exist a strictly decreasing sequence $\{s(n) : n \in N\}$ in Y which converges to p and a strictly increasing sequence $\{t(n) : n \in N\}$ in Y that converges to q. Then $J = \cup \{[s(n), t(n)] : n \in N\}$ is an expression that satisfies the above conditions (1) and (2).

Theorem 3. *Let Y be a GO-space. If there exists a dense subspace X of Y such that every relatively* *discrete subspace of X is an F_σ-subset of Y, then Y is perfect.*

Proof. We show that Y is perfect when X satisfies the condition stated in the theorem. Let U be an open subset of Y. Since Y is a GO-space, U is expressed as the union of open convex components $\{U(\alpha) : \alpha < \kappa\}$, that is, $U = \cup\{U(\alpha) : \alpha < \kappa\}$. Since X is dense in Y, $U(\alpha) \cap X$ is not empty for each $\alpha < \kappa$. Choose a point $z(\alpha) \in U(\alpha) \cap X$ and let $Z = \{z(\alpha) : \alpha < \kappa\}$. Since Z is a relatively discrete subspace of X, Z is an F_σ-subset of Y, say $Z = \cup\{Z(n) : n \in N\}$, where $Z(n)$ is closed in Y for each $n \in N$. By Lemma 3, for each $\alpha < \kappa$ we can write $U(\alpha) = \cup\{C(\alpha, m) : m \in N\}$, where

(1) $C(\alpha, m)$ is closed, convex in Y for each $m \in N$,

(2) $C(\alpha, 1) \subset C(\alpha, 2) \subset \dots \subset C(\alpha, m) \subset \dots$,

(3) $z(\alpha) \in C(\alpha, m)$ for each $m \in N$.

Let $K(n, m) = \cup\{C(\alpha, m) : z(\alpha) \in Z(n)\}$. Notice that $C(\alpha, m) \cap C(\beta, m) = \phi$ if $\alpha \neq \beta$ since $C(\alpha, m) \subset U(\alpha)$, $C(\beta, m) \subset U(\beta)$ and $U(\alpha) \cap U(\beta) = \phi$. We show that $K(n, m)$ is a closed subset of Y. Let $x \in Y - K(n, m)$. If $x \in U$, then $x \in U(\alpha)$ for some $\alpha < \kappa$. Suppose that $z(\alpha)$ does not belong to $Z(n)$. Then $U(\alpha)$ is a neighborhood of x such that $U(\alpha) \cap K(n, m) = \phi$. If $z(\alpha) \in Z(n)$, then x does not belong to $C(\alpha, m)$ because $Z(n) \subset K(n, m)$. Therefore, $U(\alpha) - C(\alpha, m)$ is a neighborhood of x and satisfies $(U(\alpha) - C(\alpha, m)) \cap K(n, m) = \phi$. Now let $x \in Y - U$. Since x is not in $Z(n)$, there exists a convex open neighborhood $W(x)$ of x such that $W(x) \cap Z(n) = \phi$ because $Z(n)$ is closed in Y. Since Y is a GO-space, $W(x)$ does not meet more than two elements of $\{C(\alpha, m) : z(\alpha) \in Z(n)\}$. Then $W = W(x) - \cup\{C(\alpha, m) : z(\alpha) \in Z(n), W(x) \cap C(\alpha, m) \neq \phi\}$ is a neighborhood of x such that $W \cap K(n, m) = \phi$. Hence $K(n, m)$ is a closed subset of Y and $U = \cup\cup\{K(n, m) : n \in N, m \in N\}$ is an F_σ-subset of Y. Therefore, Y is perfect.

The following is a lemma in [BHL] that was used to show Theorem 3.

Corollary 3. *Let X be a GO-space. If every relatively discrete subset of X is an F_σ-subset of X, then X is perfect.*

Proof. It is only necessary to let $Y = X$ in Theorem 3.

Definition 8. Let $X = (X, \tau, <)$ be a GO-space, where τ is a topology given on X. Let $R = \{x \in X : [x, \rightarrow[\in \tau - \lambda\}$ and $L = \{x \in X :]\leftarrow, x] \in \tau - \lambda\}$ (see also Definition 5). A minimal linearly ordered extension $L(X)$ of X is defined as follows [BHM]: $L(X) = (X \times \{0\}) \cup (R \times \{-1\}) \cup (L \times \{1\})$ $(\subset X \times \{-1, 0, 1\})$ is a linearly ordered topological space with a lexicographic order.

Corollary 4. *Let X be a GO-space and $L(X)$ be the minimal linearly ordered extension of X. Then $L(X)$ is perfect if X satisfies the following condition : Every relatively discrete subspace of X is an F_σ-subset of $L(X)$.*

Proof. By the definition of $L(X)$, X is a dense subspace of $L(X)$. The result easily follows from Theorem 3.

5. Counterexamples

Theorem 3 is not true for a general space Y. That means the assumption that Y is a GO-space is needed. See Examples 1 and 2, below. And also it is necessary that the existence of a dense subspace of Y that satisfies the condition in Theorem 3 as is shown in Example 3 below.

Example 1. Let $Y = [0, 1]^c$ be the cartesian product of c-many copies of the unit interval $[0, 1]$, where c denotes the cardinality of the set of real numbers. Since Y is separable [E, p.135], there exists a countable dense subset X of Y. It is clear that X and Y satisfy the condition of Theorem 3. But, Y is not perfect. To see that, let $\mathbf{0}$ be a point of Y such that $\mathbf{0}(t) = 0$ for each $t < c$. A neighborhood of $\mathbf{0}$ is of the form $\{V(t_1) \times V(t_2) \times ... \times V(t_n)\} \times \Pi \{I(t_i) : i$ is distinct from $1, 2, ..., n\}$, where $V(t_i)$ is a neighborhood of 0 in $[0, 1]$ and $I(t_i) = [0, 1]$. Hence $\mathbf{0}$ is not a G_δ-subset of Y. Therefore, Y is not perfect.

This example shows that Lemma 2 can not be generalized to general topological spaces, because $[0, 1]^c$ is not first-countable.

Example 2. Let $Y = \beta(N)$ be the Stone-Čech compactification of N, where N denotes the set of natural numbers. For a completely regular space X, the Stone-Čech compactification of X, $\beta(X)$, is defined as follows: Let $C(X)$ denote the set of all continuous functions from X to $I = [0, 1]$. When $f : X \to I$ belongs to $C(X)$, I is denoted by $I(f)$. Let $F : X \to \Pi \{I(f) : f \in C(X)\}$ be the map defined by $F(x)(f) = f(x)$, then F is an embedding. Let $\beta(X) = \mathrm{Cl}\, F(X)$. Then X is a dense subset of $\beta(X)$. This is called the Stone-Čech compactification of X. Since N is a countable dense subset of $\beta(N)$, $X = N$ and $Y = \beta(N)$ satisfy the condition of Theorem 3. Since a point $x \in Y - X$ is not a G_δ-subset of Y, Y is not perfect.

This example also shows that Lemma 2 does not hold unless Y is a GO-space.

Example 3. Let Y be the Michael line, that is, $Y = R$ is topologized as follows: Each irrational point is isolated and each rational point has Euclidean neighborhoods. Then $X = Q$, the set of rational numbers, is not a dense subspace of Y. The remaining conditions are fulfilled, but Y is not perfct. Hence the “denseness” is needed in Theorem 3.

Example 4. Let Y be the Souslin line, that is, Y is a linearly ordered topological space such that Y satisfies the countable chain condition (= the cardinality of pairwise disjoint open subsets of Y is countable) and Y is not separable. Then every dense subspace X of Y satisfies the condition :

Every relatively discrete subspace of X is an F_σ -subset of Y. To see this, let Z be a relatively discrete subspace of X. Since a GO-space is hereditary collectionwise normal, there exists an open neighborhood $U(z)$ of z in Y for each $z \in Z$ such that $\{U(z) : z \in Z\}$ is a pairwise disjoint collection. Hence Z is countable. Therfore, Z is an F_σ -subset of Y as required. Hence Y is perfect. In fact, Y is hereditarily Lindelöf.

6. Applications

Proposition 3. *Let X be a dense metrizable subspace of a GO-space Y such that the following condition is satisfied : Every relatively discrete subspace of X is an F_σ -subset of Y. Then Y has a σ -closed discrete dense subset, that is, Y has Property I.*

Proof. Theorem 3 shows that Y is perfect. By Propositions 1 and 2, Y has Property I. Also this proposition follows from a result in [BLP]: A perfect GO-space that has a dense metrizable subspace has a σ-closed discrete dense subset.

Proposition 4. Let Y be a GO-space that has a point-countable base and X a dense metrizable subspace of Y such that the following condition is satisfied: Every relatively discrete subspace of X is an F_σ-subset of Y. Then Y is metrizable.

Proof. As is shown in Proposition 3, Y has a σ-closed discrete dense subset. In [BL1], H. R. Bennett and D. J. Lutzer proved that a GO-space that has a σ-closed discrete dense subset and has a point-countable base is metrizable.

Proposition 5. *Let Y be a Souslin line with a point-countable base. Then there is no dense metrizable subspace of Y.*

Proof. Suppose that X is a dense metrizable subspace of Y. Then Y has Property II. Since Y is perfect (see Example 4), there exists a σ-closed discrete dense subset of Y by Propositions 1 and 2. Hence Y is metrizable by a theorem in [BL1] since Y has a point-countable base. This is a contradiction because Y is not metrizable.

Proposition 6. Let X be a dense metrizable subspace of a connected linearly ordered topological space Y. If every relatively discrete subspace of X is an F_σ-subset of Y, then Y is metrizable.

Proof. It follows from a theorem in [BLP] that Y has a σ-closed discrete dense subset since Y is perfect by Theorem 3 and since Y has a dense metrizable subspace. The proposition follows from a theorem of van Wouwe [vW]: A GO-space Y is metrizable if a connected linearly ordered topological space has a σ-closed discrete dense subset.

7. Questions

(1) What is the largest category of topological spaces in which Property I implies Property II?

(2) Find a condition of a topological space that is different from perfectness for which Property IV implies Property I.

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Tokyo Kasei Gakuin University
Department of Housing and Planning
2600 Aihara, Machida, Tokyo 194-0211
JAPAN